Random Walks on Clifford Algebras as Directed Hypercubes

René Schott, Stacey Staples

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Abstract

Given a Clifford algebra of arbitrary signature $\mathcal{C}\ell_{p,q}$, $p+q=n$, multiplicative random walks are induced by sequences of independent, uniformly distributed random variables taking values in the unit basis paravectors in the algebra. These walks take values in the positive and negative basis multivectors of the algebra and can be treated as random walks on directed hypercubes. Methods of walks on hypercubes are employed to develop limit theorems. These multiplicative walks are then used to induce additive walks on the Clifford algebra. Again, limit theorems are developed.

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1 Introduction

Clifford algebras, also known as geometric algebras, are higher-dimensional generalizations of complex numbers with numerous applications in physics and engineering. The n-particle fermion algebra is canonically isomorphic to the Clifford algebra $\mathcal{C}\ell_{n,n}$. The n-particle fermionic Fock space is canonically isomorphic to the Clifford algebra $\mathcal{C}\ell_{n,0}$. The Clifford algebra $\mathcal{C}\ell_{0,1}$ is isomorphic to the field of complex numbers, and $\mathcal{C}\ell_{0,2}$ is isomorphic to the quaternions. $\mathcal{C}\ell_{1,3}$ is isomorphic to the space-time algebra,

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and $\mathcal{C}\ell_{3,0}$ is the "algebra of physical space," familiar to physicists as the algebra of the Pauli spin matrices [4].

Recent applications of Clifford algebras include quantum field theory[6], computer vision[11], and automated reasoning[13]. Extending Clifford-algebraic methods to graph theory (cf. [17], [18]) opens the door to applications in theoretical computer science, symbolic dynamics, and coding theory. In particular, the graph-theoretic methods developed in [17] are used in the current work to establish limit theorems for random walks on Clifford algebras.

The group structure underlying the Clifford algebra $\mathcal{C}\ell_{p,q}$ of dimension 2^n has Cayley graph resembling the $(n+1)$-cube. Hence, results pertaining to random walks on hypercubes can be extended (with some modification) to random walks on Clifford algebras of arbitrary signature. One example of such work includes the paper by Diaconis, Graham, and Morrison [5].

In the work of Letac and Takács [12], random walks on the equivalence classes of vertex weights in the m-cube are formulated as time-homogeneous Markov chains. The spectrum of the transition probability matrix is explicitly determined with Kravchuk polynomials appearing naturally in the computation. Kravchuk matrices and related identities appearing in [12] can also be found in the work of Feinsilver and Fitzgerald [7], and their natural relationship to random walks are discussed in a paper by Feinsilver and Kocik [8].

Kravchuk polynomials are naturally connected with Clifford algebras as well. For fixed $n = p + q$, let $\phi_i(j, n)$ denote the i^{th} *Kravchuk polynomial* evaluated at j. One can show by direct computation that in the Clifford algebra $\mathcal{C}\ell_{p,q}$, the sum of the squares of the ℓ-vectors is given by

$$\sum_{|\underline{i}|=\ell} \mathbf{e}_{\underline{i}}{}^2 = (-1)^{\ell(\ell-1)/2}\phi_\ell(q, n). \tag{1.1}$$

Historically, stochastic processes on Clifford algebras have mostly been considered in the context of processes on the fermion field [1], [2], [3]. A central limit converging to "fermion quasi-free states" was developed by von Waldenfels [20], in which Clifford algebras are not assumed *a priori* but whose structure appears in the limit theorem. R.L. Hudson also developed a central limit for fermions [10].

Other examples of non-commutative central limit theorems include von Waldenfels [19], Speicher [15], and Schürmann [14]. A paper closer in approach to the current work is that of Guillotin-Plantard and Schott [9], who considered dynamic random walks on Heisenberg groups.

1.1 Notational Preliminaries

Definition 1.1. For fixed $n \geq 0$, let V be an n-dimensional vector space having orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. The 2^n-dimensional *Clifford algebra* of signature (p,q), where $p+q=n$, is defined as the associative algebra generated by the collection $\{\mathbf{e}_i\}$ along with the scalar $\mathbf{e}_\emptyset = 1 \in \mathbb{R}$, subject to the following multiplication rules:

$$\mathbf{e}_i\,\mathbf{e}_j + \mathbf{e}_j\,\mathbf{e}_i = 0 \text{ for } i \neq j, \text{ and} \tag{1.2}$$

$$\mathbf{e}_i{}^2 = \begin{cases} 1, \text{ if } 1 \leq i \leq p \\ -1, \text{ if } p+1 \leq i \leq p+q = n. \end{cases} \tag{1.3}$$

The Clifford algebra of signature (p,q) is denoted $\mathcal{C}\ell_{p,q}$.

Generally the vectors generating the algebra do not have to be orthogonal. When they are orthogonal as in the definition above, the resulting multivectors are called *blades*.

Let $[n] = \{1, , 2, \ldots, n\}$ and denote arbitrary, canonically ordered subsets of $[n]$ by underlined Roman characters. The basis elements of $\mathcal{C}\ell_{p,q}$ can then be indexed by these finite subsets by writing

$$\mathbf{e}_{\underline{i}} = \prod_{k \in \underline{i}} \mathbf{e}_k. \tag{1.4}$$

Arbitrary elements of $\mathcal{C}\ell_{p,q}$ have the form

$$u = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}}\,\mathbf{e}_{\underline{i}}, \tag{1.5}$$

where $u_{\underline{i}} \in \mathbb{R}$ for each $\underline{i} \in 2^{[n]}$.

An arbitrary element $u \in \mathcal{C}\ell_{p,q}$ is called a *paravector* if it has the form

$$u = u_\emptyset + \sum_{i=1}^{n} u_i\,\mathbf{e}_i\,. \tag{1.6}$$

In other words, a paravector is the sum of a scalar and a vector.

The *inner-product* of $u, v \in \mathcal{C}\ell_{p,q}$ is defined by

$$\langle u, v \rangle = \left\langle \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}}\,\mathbf{e}_{\underline{i}}\,, \sum_{\underline{i} \in 2^{[n]}} v_{\underline{i}}\,\mathbf{e}_{\underline{i}} \right\rangle = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}}\,v_{\underline{i}}\,. \tag{1.7}$$

Observe that for fixed multi-index $\underline{i} \in 2^{[n]}$ and arbitrary $u \in \mathcal{C}\ell_{p,q}$, $\langle u, \mathbf{e}_{\underline{i}} \rangle = u_{\underline{i}}$, the coefficient of the multivector $\mathbf{e}_{\underline{i}}$ in the canonical expansion of u. The inner product induces a *Clifford inner-product norm* by

$$\|u\|^2 = \langle u, u \rangle = \sum_{\underline{i} \in 2^{[n]}} {u_{\underline{i}}}^2. \tag{1.8}$$

2 Random walks on $\mathcal{C}\ell_{p,q}$

Given a random variable ξ, the notation $\langle \xi \rangle$ will denote the *expectation* of ξ. Given a sequence of random variables $\{\xi_N\}$, the notation $\xi_N \xrightarrow{\mathcal{D}} \psi$ denotes *convergence in distribution* to the random variable ψ. The notation $\xi_N \xrightarrow{\mathcal{P}} u$ denotes *convergence in probability* to u.

For fixed $n > 0$, let $(X_0(k), X_1(k), X_2(k), \ldots, X_n(k))$, $1 \leq k$, be a sequence of independent random variables taking values in the unit coordinate vectors $(e_i)_{0 \leq i \leq n}$ of $\mathbb{R}^{n+1}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For each $j \geq 1$, the distribution of the random vector $M_j = (X_1(j), \ldots, X_n(j))$ is

$$\mathbb{P}(M_j = e_i) = \frac{1}{n+1}, \; 1 \leq i \leq n. \tag{2.1}$$

Let $(S_k)_{1 \leq k}$ be the right random walk on $\mathcal{C}\ell_{p,q}$ induced by the sequence $(X_0(k), X_1(k), X_2(k), \ldots, X_n(k))$ via

$$S_k = (X_0(1), \ldots, X_n(1)) \cdot (X_0(2), \ldots, X_n(2)) \cdots (X_0(k), \ldots, X_n(k)) \,. \tag{2.2}$$

Hence, the sequence $(S_k)_{k \geq 1}$ is a random walk on the blades of $\mathcal{C}\ell_{p,q}$.

Walks of this form have a natural connection with classical walks on the n-dimensional hypercube (cf. [5], [17]). This connection will be exploited to prove limit theorems.

Lemma 2.1. *Fix* $\underline{i} \in 2^{[n]}$. *Then,*

$$\sum_{\substack{\ell_0 + \cdots + \ell_n = k \\ \ell_j \text{ odd when } 0 \neq j \in \underline{i},\, \ell_j \text{ even when } 0 \neq j \notin \underline{i}}} \binom{k}{\ell_0, \ldots, \ell_n} \to \frac{(n+1)^k}{2^n} \tag{2.3}$$

asymptotically as $k \to \infty$.

Proof. It is not difficult to see that the random sequence $(X_0(k), X_2(k), \ldots, X_n(k))$ induces a nearest-neighbor walk on the n-dimensional hypercube $\mathcal{Q}_n$ in which a particle moves to one of its nearest neighbors or stays fixed with equal probability. The 2^n

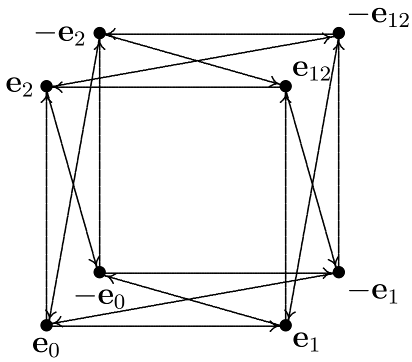

Figure 1: Graph associated with walks on $\mathcal{C}\ell_{0,2}$ visualized as a directed hypercube on 2^3 vertices.

vertices of $\mathcal{Q}_n$ are in one-to-one correspondence with binary strings of length n, and assume the walk starts from $S_0 = v_\emptyset$. Multiplication by the unit vector e_j corresponds to "flipping" the j^{th} bit.

Let $\underline{i}$ denote a multi-index corresponding to the length-n binary string $\vec{b} = (b_1\, b_2 \cdots b_n)$ via

$$b_j = \begin{cases} 1 & \text{if } j \in \underline{i} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Notice that $S_0 = v_\emptyset$ implies $S_k = v_{\underline{i}}$ if and only if each element of the multi-index $\underline{i}$ has been "flipped" an odd number of times, while each element of the complement $\underline{i}'$ has been "flipped" an even number of times. There is no restriction on ℓ_0 since it corresponds to the number of steps in which the particle stays fixed. Because the steps of the walk are independent and identically distributed, the probability of being at vertex v_i at time step k is then given by

$$\begin{aligned} \mathbb{P}(S_k = v_i) &= \frac{1}{(n+1)^k} \left|\{k\text{-step walks } v_\emptyset \to v_i\}\right| \\ &= \frac{1}{(n+1)^k} \sum_{\substack{\ell_0+\ell_1+\cdots+\ell_n=k \\ \ell_j \text{ odd when } j\in\underline{i},\, \ell_j \text{ even when } j\notin\underline{i}}} \binom{k}{\ell_1, \ldots, \ell_n}. \end{aligned} \tag{2.5}$$

The limiting distribution on the vertices of $\mathcal{Q}_n$ is known to be uniform[5]. Hence,

$$\frac{1}{(n+1)^k} \sum_{\substack{\ell_0+\cdots+\ell_n=k \\ \ell_j \text{ odd when } j\in\underline{i},\, \ell_j \text{ even when } j\notin\underline{i}}} \binom{k}{\ell_0, \ldots, \ell_n} \to \frac{1}{2^n}, \tag{2.6}$$

as $k \to \infty$. $\square$

Theorem 2.2. *Let $(S_k)_{k\geq 1}$ denote the Clifford random walk defined in (2.2). Let ψ be the uniformly distributed random variable taking values in $\{\pm\mathbf{e}_{\underline{i}}\}$, the positive and negative basis multivectors of $\mathcal{C}\ell_{p,q}$. Then,*

$$\lim_{k\to\infty} \mathbb{P}\left(S_k = \alpha\,\mathbf{e}_{\underline{i}}\right) = \frac{1}{2^{n+1}} \tag{2.7}$$

where $\alpha = \pm 1$. Further,

$$\lim_{k\to\infty} \langle S_k \rangle = 0, \tag{2.8}$$

and

$$\lim_{k\to\infty} \left|\left|\frac{S_k}{k}\right|\right| = 0. \tag{2.9}$$

Also, as $k \to \infty$

$$S_k \xrightarrow{\mathcal{D}} \psi. \tag{2.10}$$

Proof. Observe that the walk S_k satisfies

$$\mathbb{P}\left(S_k = \pm\mathbf{e}_{\underline{i}}\right) = \frac{1}{(n+1)^k} \sum_{\substack{\ell_0+\cdots+\ell_n=k \\ \ell_j \text{ odd when } j\in\underline{i},\, \ell_j \text{ even when } j\notin\underline{i}}} \binom{k}{\ell_0,\ldots,\ell_n} \to \frac{1}{2^n} \tag{2.11}$$

by Lemma 2.1. The sign $\alpha = \pm 1$ is determined by the order in which the vertices of the walk are visited, independent of the signature of the algebra. This independence is attributable to the fact that negative squares are equally represented among permutations of paths. Hence, $\alpha = \operatorname{sgn}(\varsigma_k)$ where $\varsigma_k \in \mathbf{S}_n$ is the permutation that canonically orders the indices of the sequence S_k (excluding the 0 index, which corresponds to the unit scalar). Exactly half of these permutations lead to $\alpha = 1$ and half to $\alpha = -1$.

The expectation of the sequence (S_k) is found by straightforward calculation:

$$\begin{aligned}\lim_{k\to\infty} \langle S_k \rangle = \lim_{k\to\infty} &\sum_{\underline{i}\in 2^{[n]}} \sum_{\alpha=\pm 1} \mathbb{P}\left(S_k = \alpha\,\mathbf{e}_{\underline{i}}\right) \alpha\,\mathbf{e}_{\underline{i}} \\ &\sum_{\underline{i}\in 2^{[n]}} \sum_{\alpha=\pm 1} \lim_{k\to\infty} \mathbb{P}\left(S_k = \alpha\,\mathbf{e}_{\underline{i}}\right) \alpha\,\mathbf{e}_{\underline{i}} \\ &= \sum_{\underline{i}\in 2^{[n]}} \sum_{\alpha=\pm 1} \frac{1}{2^{n+1}} \alpha\,\mathbf{e}_{\underline{i}} = \frac{1}{2^{n+1}} \left(\sum_{\underline{i}\in 2^{[n]}} \mathbf{e}_{\underline{i}} - \sum_{\underline{i}\in 2^{[n]}} \mathbf{e}_{\underline{i}} \right) = 0.\end{aligned} \tag{2.12}$$

Since (S_k) is a sequence of unit basis multivectors in $\mathcal{C}\ell_{p,q}$, the limit of the L^2 norm is

$$\lim_{k\to\infty} \left|\left|\frac{S_k}{k}\right|\right|^2 = \lim_{k\to\infty} \frac{1}{k} = 0. \tag{2.13}$$

Finally, the limiting distribution is clearly uniform on the collection $\{\alpha\, \mathbf{e}_{\underline{i}}\}$, $\alpha = \pm 1$, $\underline{i} \in 2^{[n]}$. $\square$

2.1 The induced additive walk on $\mathcal{C}\ell_{p,q}$

For each $N > 0$, assume the collection $\{S_1, \ldots, S_N\}$ is independent. That is, not only is the collection $\{(X_1(k), \ldots, X_n(k))\}_{k\geq 1}$ independent, but the collection of products $\{\prod_{k=1}^{N}(X_1(k), \ldots, X_n(k))\}_{N>0}$ is assumed to be independent. Define the Clifford-valued walk

$$\Xi_N = \sum_{k=1}^{N} S_k. \tag{2.14}$$

The goal is to prove a law of large numbers and a central limit theorem for the walk $(\Xi_N)_{N>0}$. To this end, a useful identity is taken from [16].

Lemma 2.3.

$$\left(\sum_{i=1}^{n} a_i\, \mathbf{e}_i\right)^k = \begin{cases} \left(\sum_{i=1}^{p} {a_i}^2 - \sum_{j=p+1}^{n} {a_j}^2\right)^{k/2} & \text{if } k \equiv 0 \pmod 2 \\ \left(\sum_{i=1}^{p} {a_i}^2 - \sum_{j=p+1}^{n} {a_j}^2\right)^{(k-1)/2} \sum_{i=1}^{n} a_i\, \mathbf{e}_i & \text{if } k \equiv 1 \pmod 2. \end{cases} \tag{2.15}$$

The next result shows that the expectation of the walk Ξ_N is *paravector*-valued.

Proposition 2.4. *Let* $(\Xi_N)_{N>0}$ *be the walk defined in* (2.14)*. For any positive integer* N*,*

$$\langle \Xi_N \rangle = \sum_{k=1}^{N} \frac{1}{n^k} \sum_{\substack{0\leq \ell \leq k \\ k-\ell \text{ even}}} (p-q)^{\frac{k-\ell}{2}} \left(\binom{k}{\ell} + \binom{k}{\ell+1} \sum_{i=1}^{n} \mathbf{e}_i\right). \tag{2.16}$$

Proof. By linearity of expectation,

$$\langle \Xi_N \rangle = \left\langle \sum_{k=1}^{N} S_k \right\rangle = \sum_{k=1}^{N} \langle S_k \rangle. \tag{2.17}$$

Applying Lemma 2.3, the expectation of S_k is given by

$$\begin{aligned}
\langle S_k\rangle &= \left(\frac{1}{n}+\frac{1}{n}\sum_{i=1}^{n}\mathbf{e}_i\right)^{k} = \sum_{\ell=0}^{k}\binom{k}{\ell}\frac{1}{n^{\ell}}\frac{1}{n^{k-\ell}}\left(\sum_{i=1}^{n}\mathbf{e}_i\right)^{k-\ell}\\
&= \frac{1}{n^k}\sum_{\substack{0\le\ell\le k\\ k-\ell \text{ even}}}\binom{k}{\ell}(p-q)^{\frac{k-\ell}{2}} + \frac{1}{n^k}\sum_{\substack{0\le\ell\le k\\ k-\ell \text{ odd}}}\binom{k}{\ell}(p-q)^{\frac{k-\ell-1}{2}}\sum_{i=1}^{n}\mathbf{e}_i\\
&= \frac{1}{n^k}\sum_{\substack{0\le\ell\le k\\ k-\ell \text{ even}}}(p-q)^{\frac{k-\ell}{2}}\left(\binom{k}{\ell}+\binom{k}{\ell+1}\sum_{i=1}^{n}\mathbf{e}_i\right). \qquad (2.18)
\end{aligned}$$

□

Corollary 2.5. *In the Clifford algebra $\mathcal{C}\ell_{n,n}$, canonically isomorphic to the n-particle fermion algebra, $\langle \Xi_N\rangle = 0$ for all $N > 0$.*

The goal is to prove a law of large numbers and a central limit theorem for the walk $(\Xi_N)_{N\geq 1}$. In order to develop these, some additional tools are borrowed from [17].

Definition 2.6. For fixed $n > 0$, the algebra $\mathcal{C}\ell_n{}^{\mathrm{sym}}$ is defined as the 2^n-dimensional associative algebra generated by the elements ε_i for $1 \leq i \leq n$ along with the scalar $\varepsilon_\emptyset = 1 \in \mathbb{R}$, subject to the following multiplication rules:

$$\varepsilon_i\,\varepsilon_j = \begin{cases}\varepsilon_j\,\varepsilon_i & \text{if } i\neq j\\ 1 & \text{otherwise.}\end{cases} \qquad (2.19)$$

Basis elements of $\mathcal{C}\ell_n{}^{\mathrm{sym}}$ can be indexed by canonically-ordered subsets of $[n]$ so that arbitrary elements have the form

$$u = \sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}\,\varepsilon_{\underline{i}}. \qquad (2.20)$$

By the properties of Clifford multiplication, given any $\underline{i},\underline{j} \in 2^{[n]}$,

$$\varepsilon_{\underline{i}}\,\varepsilon_{\underline{j}} = \varepsilon_{\underline{i}\triangle\underline{j}}\,, \qquad (2.21)$$

where $\triangle = (\underline{i}\cup\underline{j})\setminus(\underline{i}\cap\underline{j})$ is the symmetric difference of $\underline{i}$ and $\underline{j}$.

It is evident that the generators of ε_i of $\mathcal{C}\ell_n{}^{\mathrm{sym}}$ generate a multiplicative group Σ_n isomorphic to the group generated by reflections across orthogonal hyperplanes in the real vector space $\mathbb{R}^n$, for these also satisfy $R_i\,R_j = R_j\,R_i$ and ${R_i}^2 = \mathrm{id}$. It is equally evident that $\Sigma_n \cong (2^{[n]}, \triangle)$, the group consisting of the power set of $[n] = \{1, 2, \dots, n\}$ with the set symmetric difference operator. These groups are also isomorphic to the additive abelian group $\underbrace{\mathbb{Z}_2 + \cdots + \mathbb{Z}_2}_{n\text{-times}}$. Because of these isomorphisms, it should be clear that the Cayley graph of Σ_n is the n-dimensional hypercube $\mathcal{Q}_n$.

Vertices of $\mathcal{Q}_n$ can be canonically labeled with binary strings of length n. Bit strings of length n are associated with unit basis multivectors of ${\mathcal{C}\ell_n}^{\mathrm{sym}}$. The product of two such multivectors corresponds to the logical "exclusive or" of the associated bit strings.

Remark 2.7. The algebra ${\mathcal{C}\ell_n}^{\mathrm{sym}}$ is an abelian subalgebra of $\mathcal{C}\ell_{n,n}$.

The following theorem, proved in [17], will be of use.

Theorem 2.8. *Let Y be a random variable taking values in $\{0, 1, 2, \dots, n\}$ with probabilities $p_i = \Pr\{Y = i\}$ for each $0 \le i \le n$, and let $\{Y_k\}_{k>0}$ be the sequence of independent random variables obtained from repeated observations of Y. Let $\xi_k \in {\mathcal{C}\ell_n}^{\mathrm{sym}}$ represent the distribution on the vertices of $\mathcal{Q}_n$ at time step $k \ge 0$ corresponding to the random walk induced by the sequence $\{Y_k\}$. Then for $k > 0$ and real parameter $t \ne 0$,*

$$\xi_k = k! \cdot \xi_0\, e^{t\,p_0} \prod_{i=1}^{n} (\cosh(t\,p_i) + \sinh(t\,p_i)\,\varepsilon_i)\Big|_{t^k}. \tag{2.22}$$

Here the notation $\big|_{t^k}$ on the right-hand side represents the Clifford-valued coefficient of t^k in the formal power series expansion of the product.

Theorem 2.9. *Let $(S_k)_{k\ge1}$ denote the Clifford random walk defined in* (2.2)*. Then for fixed k and $\alpha = \pm1$, the distribution of S_k is given by the generating function*

$$\mathbb{P}\left(S_k = \alpha\,\mathbf{e}_{\underline{i}}\right) = \frac{k!}{2} e^{\frac{t}{n+1}} \cosh^{n-|\underline{i}|}\left(\frac{t}{n+1}\right) \sinh^{|\underline{i}|}\left(\frac{t}{n+1}\right)\Big|_{t^k}. \tag{2.23}$$

Proof. In the case $p_i = \dfrac{1}{n+1}$ for $0 \le i \le n$, one obtains from Theorem 2.8

$$\xi_k = k! \cdot \xi_0\, e^{t/(n+1)} \prod_{i=1}^{n} \left(\cosh\left(\frac{t}{n+1}\right) + \sinh\left(\frac{t}{n+1}\right)\,\varepsilon_i\right)\Big|_{t^k}. \tag{2.24}$$

Letting $\xi_0 = \varepsilon_0$, and utilizing the correspondence with walks on $\mathcal{C}\ell_{p,q}$ induced by paravector sequences,

$$\mathbb{P}(S_k = \alpha\,\mathbf{e}_{\underline{i}}) = \frac{1}{2}\left\langle \xi_k, \varepsilon_{\underline{i}} \right\rangle, \tag{2.25}$$

so that for $\alpha = \pm 1$,

$$\mathbb{P}(S_k = \alpha\, \mathbf{e}_{\underline{i}}) = \frac{k!}{2} \left\langle e^{\frac{t}{n+1}} \prod_{i=1}^{n} \left(\cosh\left(\frac{t}{n+1}\right) + \sinh\left(\frac{t}{n+1}\right) \varepsilon_i \right)\Big|_{t^k}, \varepsilon_{\underline{i}} \right\rangle. \tag{2.26}$$

Expanding the product, the coefficient of the multivector $\varepsilon_{\underline{i}}$ depends only on the cardinality of the index $\underline{i}$, since $p_i = \frac{1}{n+1}$ for all $0 \leq i \leq n$. $\square$

Corollary 2.10. *For $k \geq 1$ and $\alpha = \pm 1$,*

$$\mathbb{P}\left(S_k = \alpha\right) = \frac{k! e^{\frac{t}{n+1}}}{2} \cosh^n\left(\frac{t}{n+1}\right)\Big|_{t^k}, \tag{2.27}$$

and for $1 \leq i \leq n$,

$$\mathbb{P}\left(S_k = \alpha\, \mathbf{e}_i\right) = \frac{k! e^{\frac{t}{n+1}}}{2} \cosh^{n-1}\left(\frac{t}{n+1}\right) \sinh\left(\frac{t}{n+1}\right)\Big|_{t^k}. \tag{2.28}$$

Theorem 2.11. *Let $(\Xi_N)_{N>0}$ be the walk defined in (2.14) on $\mathcal{C}\ell_{p,q}$, $p+q>0$, and let $\vec{x} = \mathbf{e}_1 + \cdots + \mathbf{e}_n$. Then the following limit exists if and only if $(p,q) \notin \{(1,0),(0,1),(2,0)\}$:*

$$\tilde{\Xi} = \lim_{N\to\infty} \langle \Xi_N \rangle = \mu_0 + \mu_1 \vec{x}, \tag{2.29}$$

where

$$\mu_0 = \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{n^k} \sum_{\substack{0\leq \ell\leq k\\ k-\ell \text{ even}}} (p-q)^{\frac{k-\ell}{2}} \binom{k}{\ell}, \text{ and} \tag{2.30}$$

$$\mu_1 = \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{n^k} \sum_{\substack{0\leq \ell\leq k\\ k-\ell \text{ even}}} (p-q)^{\frac{k-\ell}{2}} \binom{k}{\ell+1}. \tag{2.31}$$

Proof. Begin by noting that

$$\tilde{\Xi} = \lim_{N\to\infty} \langle \Xi_N \rangle = \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{n^k} \sum_{\substack{0\leq \ell\leq k\\ k-\ell \text{ even}}} (p-q)^{\frac{k-\ell}{2}} \left(\binom{k}{\ell} + \binom{k}{\ell+1} \sum_{i=1}^{n} \mathbf{e}_i \right), \tag{2.32}$$

so that when the limit exists, μ_0 and μ_1 are the stated limits.

The theorem is proved in six steps:

(i.) The series converges when $p = q \geq 1$.

When $p = q$, $\tilde{\Xi} = 0$ by Corollary 2.5.

To handle all other cases, consider the series

$$\sum_{k=1}^{\infty} \frac{1}{(p+q)^k} \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} (p-q)^{\ell} \binom{k}{2\ell}. \tag{2.33}$$

(ii.) The series converges absolutely when $p + q > 4$.

First, to see that the series converges absolutely when $p + q > 4$, consider the following:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{(p+q)^k} \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} |p-q|^{\ell} \binom{k}{2\ell} &\leq \sum_{k=1}^{\infty} \frac{1}{(p+q)^k} \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} |p+q|^{\ell} \binom{k}{2\ell} \\
&= \sum_{k=1}^{\infty} \frac{1}{(p+q)^k} \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} (p+q)^{\ell} \binom{k}{2\ell} \\
&\leq \sum_{k=1}^{\infty} \frac{1}{(p+q)^k} (p+q)^{\frac{k}{2}} \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2\ell} \\
&\leq \sum_{k=1}^{\infty} \frac{1}{(p+q)^k} (p+q)^{\frac{k}{2}} 2^{k-1} \\
&= \sum_{k=1}^{\infty} \frac{2^k}{2(p+q)^{k/2}} = \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{2}{\sqrt{p+q}} \right)^k.
\end{aligned} \tag{2.34}$$

This series converges provided $\sqrt{p+q} > 2$.

(iii.) The series converges absolutely when $\dfrac{\sqrt{|p-q|}}{p+q} < 1/2$.

Consider the series comparison:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{(p+q)^k} \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} |p-q|^{\ell} \binom{k}{2\ell} &\leq \sum_{k=1}^{\infty} \frac{1}{(p+q)^k} |p-q|^{\frac{k}{2}} \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2\ell} \\
&\leq \sum_{k=1}^{\infty} \frac{1}{(p+q)^k} |p-q|^{\frac{k}{2}} 2^{k-1} \\
&= \sum_{k=1}^{\infty} \frac{2^k \sqrt{|p-q|}^k}{2(p+q)^k} = \frac{1}{2} \sum_{k=1}^{\infty} \left(2\sqrt{|p-q|} p + q \right)^k,
\end{aligned} \tag{2.35}$$

which converges provided $\dfrac{2\sqrt{|p-q|}}{p+q} < 1$. This proves convergence in $\mathcal{C}\ell_{1,3}$, $\mathcal{C}\ell_{3,1}$, $\mathcal{C}\ell_{1,2}$, and $\mathcal{C}\ell_{2,1}$.

(iv.) The series converges in $\mathcal{C}\ell_{0,3}$, $\mathcal{C}\ell_{3,0}$, $\mathcal{C}\ell_{0,4}$, and $\mathcal{C}\ell_{4,0}$.

Next, convergence of the series in $\mathcal{C}\ell_{3,0}$ will imply absolute convergence in $\mathcal{C}\ell_{0,3}$.

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(p+q)^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(p-q)^\ell\binom{k}{2\ell} &= \sum_{k=1}^{\infty}\frac{1}{3^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}3^\ell\binom{k}{2\ell}\\ &= \sum_{k=1}^{\infty}\frac{1}{3^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(\sqrt{3})^{2\ell}\binom{k}{2\ell}\\ &\leq \sum_{k=1}^{\infty}\frac{1}{3^k}\sum_{\ell=0}^{k}(\sqrt{3})^{\ell}\binom{k}{\ell} = \sum_{k=1}^{\infty}\left(\frac{1+\sqrt{3}}{3}\right)^k. \end{aligned} \tag{2.36}$$

Similarly, the series converges in $\mathcal{C}\ell_{4,0}$ and $\mathcal{C}\ell_{0,4}$.

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(p+q)^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(p-q)^\ell\binom{k}{2\ell} &= \sum_{k=1}^{\infty}\frac{1}{4^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}4^\ell\binom{k}{2\ell}\\ &= \sum_{k=1}^{\infty}\frac{1}{4^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}2^{2\ell}\binom{k}{2\ell} \leq \sum_{k=1}^{\infty}\frac{1}{4^k}\sum_{\ell=0}^{k}2^{\ell}\binom{k}{\ell} = \sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^k. \end{aligned} \tag{2.37}$$

(v.) The series converges in $\mathcal{C}\ell_{0,2}$.

Convergence in $\mathcal{C}\ell_{0,2}$ requires a different approach. In particular, consider

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(p+q)^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(p-q)^\ell\binom{k}{2\ell} &= \sum_{k=1}^{\infty}\frac{1}{2^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(-2)^\ell\binom{k}{2\ell}\\ &= \sum_{k=1}^{\infty}\frac{1}{2^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(i\sqrt{2})^{2\ell}\binom{k}{2\ell}. \end{aligned} \tag{2.38}$$

It is apparent that

$$\left(\frac{1+i\sqrt{2}}{2}\right)^k = \frac{1}{2^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(i\sqrt{2})^{2\ell}\binom{k}{2\ell} + \frac{1}{2^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}}(i\sqrt{2})^{2\ell+1}\binom{k}{2\ell+1}, \tag{2.39}$$

which implies

$$\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(i\sqrt{2})^{2\ell}\binom{k}{2\ell} = \mathrm{Re}\left[(1+i\sqrt{2})^k\right]. \tag{2.40}$$

Convergence of the series $\sum_{k=1}^{\infty}\mathrm{Re}\left[(1+i\sqrt{2})^k\right]$ is now guaranteed by the inequality $\left\|\frac{1+i\sqrt{2}}{2}\right\| = \frac{\sqrt{3}}{2} < 1$. An identical argument shows convergence of the imaginary part, which is necessary for the paravector coefficient of the expectation.

(vi.) The series diverges in $\mathcal{C}\ell_{1,0}$, $\mathcal{C}\ell_{0,1}$, and $\mathcal{C}\ell_{2,0}$.

In $\mathcal{C}\ell_{1,0}$, one has the divergent series

$$\sum_{k=1}^{\infty}\frac{1}{(p+q)^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(p-q)^\ell\binom{k}{2\ell} = \sum_{k=1}^{\infty}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2\ell} = \sum_{k=1}^{\infty}2^{k-1}. \tag{2.41}$$

In $\mathcal{C}\ell_{0,1}$, one has the series

$$\sum_{k=1}^{\infty}\frac{1}{(p+q)^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(p-q)^\ell\binom{k}{2\ell} = \sum_{k=1}^{\infty}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^\ell\binom{k}{2\ell}, \tag{2.42}$$

which diverges because the sequence of partial sums $S_k = \sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^\ell\binom{k}{2\ell}$ diverges as $k \to \infty$.

In $\mathcal{C}\ell_{2,0}$, one has the series

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(p+q)^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}(p-q)^\ell\binom{k}{2\ell} &= \sum_{k=1}^{\infty}\frac{1}{2^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}2^\ell\binom{k}{2\ell}\\ &\geq \sum_{k=1}^{\infty}\frac{1}{2^k}\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2\ell} = \sum_{k=1}^{\infty}\frac{1}{2},\end{aligned} \tag{2.43}$$

which clearly diverges.

□

Example 2.12. Because $\langle \Xi_N \rangle$ is paravector-valued for all $N > 0$, an easy visualization is given by plotting points in $\mathbb{R}^2$. The x-coordinate represents the scalar coefficient, and the y-coordinate represents the coefficient of $\vec{x} = \sum_{i=1}^{n} \mathbf{e}_i$. See Figure 2.

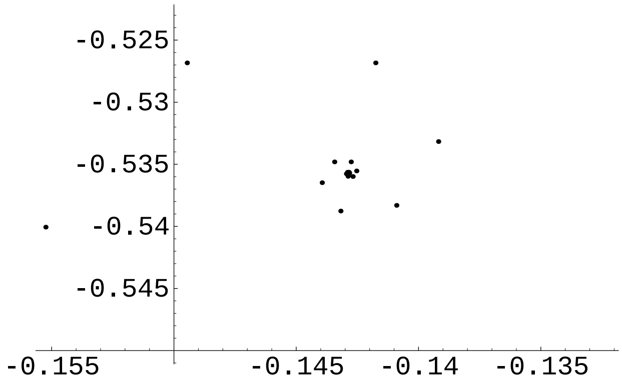

Figure 2: Expectation of Ξ_N in $\mathcal{C}\ell_{0,3}$, $1 \le N \le 32$.

Example 2.13. By direct Mathematica computation, the expectation $\langle \Xi_N \rangle$ in $\mathcal{C}\ell_{0,2}$ is seen to converge to $\tilde{\tilde{\Xi}} = -\frac{1}{3} - \frac{2}{3}\vec{x}$ (see Figure 3).

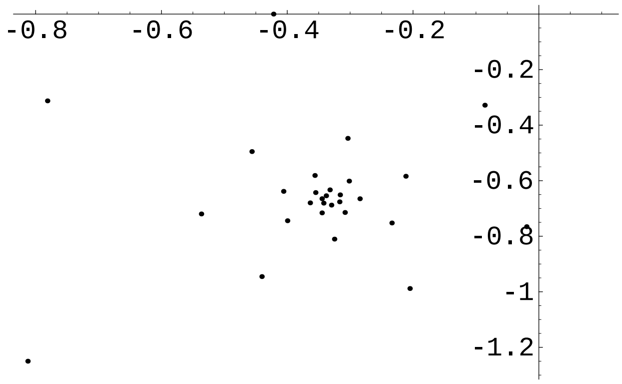

Figure 3: Expectation of Ξ_N in $\mathcal{C}\ell_{0,2}$, $1 \le N \le 32$.

Theorem 2.14. *Let $(\Xi_N)_{N>0}$ be the walk defined in (2.14) on $\mathcal{C}\ell_{p,q}$, $p + q > 0$, $(p, q) \notin \{(1, 0), (0, 1), (2, 0)\}$. Let $\tilde{\tilde{\Xi}} = \mu_0 + \mu_1\vec{x}$ as in Theorem 2.11. Then there exists a mean-zero Clifford-valued random variable Ψ of variance 1 such that as $N \to \infty$,*

$$\frac{\Xi_N - \tilde{\tilde{\Xi}}}{\sigma_N} \xrightarrow{\mathcal{D}} \Psi, \tag{2.44}$$

where the standard deviation σ_N is determined by

$$
\begin{aligned}
{\sigma_N}^2 = \sum_{k=1}^{N} \frac{\partial^k}{\partial t^k} & \left[e^{\frac{t}{n+1}} \sum_{\ell=2}^{n} \cosh^{n-\ell}\left(\frac{t}{n+1}\right) \sinh^{\ell}\left(\frac{t}{n+1}\right) \right]_{t=0} \\
& + ({\mu_0}^2 + 1) \sum_{k=1}^{N} \frac{\partial^k}{\partial t^k} \left[e^{\frac{t}{n+1}} \cosh^{n}\left(\frac{t}{n+1}\right) \right]_{t=0} \\
& + n({\mu_1}^2 + 1) \sum_{k=1}^{N} \frac{\partial^k}{\partial t^k} \left[e^{\frac{t}{n+1}} \cosh^{n-1}\left(\frac{t}{n+1}\right) \sinh\left(\frac{t}{n+1}\right) \right]_{t=0}. \qquad (2.45)
\end{aligned}
$$

Proof. By Theorem 2.11, the mean value $\tilde{\Xi}$ exists. Clearly,

$$
\lim_{N\to\infty} \left\langle \Xi_N - \tilde{\Xi} \right\rangle = 0. \qquad (2.46)
$$

The variance ${\sigma_N}^2$ of $\Xi_N - \tilde{\Xi}$ is computed using Theorem 2.9 and Corollary 2.10.

$$
\begin{aligned}
\left\langle \|\Xi_N - \tilde{\Xi}\|^2 \right\rangle &= \sum_{k=1}^{N} \sum_{\substack{\underline{i}\in 2^{[n]} \\ \alpha=\pm 1}} \mathbb{P}\left(S_k = \alpha\, \mathbf{e}_{\underline{i}}\right) \left\| S_k - \left\langle \tilde{\Xi}, \mathbf{e}_{\underline{i}} \right\rangle \right\|^2 \\
&= \sum_{k=1}^{N} \left[\sum_{\substack{|\underline{i}|>1 \\ \alpha=\pm 1}} \mathbb{P}\left(S_k = \alpha\, \mathbf{e}_{\underline{i}}\right) \|S_k\|^2 + \sum_{\substack{0\le i\le n \\ \alpha=\pm 1}} \mathbb{P}\left(S_k = \alpha\, \mathbf{e}_{i}\right) \left\| S_k - \left\langle \tilde{\Xi}, \mathbf{e}_{i} \right\rangle \right\|^2 \right] \\
&= \sum_{k=1}^{N} \sum_{\substack{|\underline{i}|>1 \\ \alpha=\pm 1}} \mathbb{P}\left(S_k = \alpha\, \mathbf{e}_{\underline{i}}\right) + \sum_{k=1}^{N} \sum_{\alpha=\pm 1} \mathbb{P}\left(S_k = \alpha\right) \left\| \alpha - \left\langle \tilde{\Xi}, \mathbf{e}_{\emptyset} \right\rangle \right\|^2 \\
&\qquad + \sum_{k=1}^{N} \sum_{\substack{1\le i\le n \\ \alpha=\pm 1}} \mathbb{P}\left(S_k = \alpha\, \mathbf{e}_{i}\right) \left\| \alpha - \left\langle \tilde{\Xi}, \mathbf{e}_{i} \right\rangle \right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{N} \sum_{\substack{2\le \ell \le n \\ \alpha=\pm 1}} \frac{1}{2} k! e^{\frac{t}{n+1}} \cosh^{n-\ell}\left(\frac{t}{n+1}\right) \sinh^{\ell}\left(\frac{t}{n+1}\right)\Big|_{t^k} \\
&\quad + \sum_{k=1}^{N} \sum_{\alpha=\pm 1} \frac{1}{2} k! e^{\frac{t}{n+1}} \cosh^{n}\left(\frac{t}{n+1}\right) \left\|\alpha - \left\langle \tilde{\Xi}, \mathbf{e}_{\emptyset}\right\rangle\right\|^2\Big|_{t^k} \\
&\quad + \sum_{k=1}^{N} \sum_{\substack{1\le i \le n \\ \alpha=\pm 1}} \frac{1}{2} k! e^{\frac{t}{n+1}} \cosh^{n-1}\left(\frac{t}{n+1}\right) \sinh\left(\frac{t}{n+1}\right) \left\|\alpha - \left\langle \tilde{\Xi}, \mathbf{e}_{i}\right\rangle\right\|^2\Big|_{t^k} \\
&= \sum_{k=1}^{N} \sum_{\ell=2}^{n} k! e^{\frac{t}{n+1}} \cosh^{n-\ell}\left(\frac{t}{n+1}\right) \sinh^{\ell}\left(\frac{t}{n+1}\right)\Big|_{t^k} \\
&\quad + \sum_{k=1}^{N} \frac{1}{2} k! e^{\frac{t}{n+1}} \cosh^{n}\left(\frac{t}{n+1}\right) \left((\mu_0-1)^2 + (\mu_0+1)^2\right)\Big|_{t^k} \\
&\quad + \sum_{k=1}^{N} \sum_{i=1}^{n} \frac{1}{2} k! e^{\frac{t}{n+1}} \cosh^{n-1}\left(\frac{t}{n+1}\right) \sinh\left(\frac{t}{n+1}\right) \left((\mu_1-1)^2 + (\mu_1+1)^2\right)\Big|_{t^k} \\
&= \sum_{k=1}^{N} k! e^{\frac{t}{n+1}} \sum_{\ell=2}^{n} \cosh^{n-\ell}\left(\frac{t}{n+1}\right) \sinh^{\ell}\left(\frac{t}{n+1}\right)\Big|_{t^k} \\
&\quad + ({\mu_0}^2+1) \sum_{k=1}^{N} k! e^{\frac{t}{n+1}} \cosh^{n}\left(\frac{t}{n+1}\right)\Big|_{t^k} \\
&\quad + ({\mu_1}^2+1) \sum_{k=1}^{N} n k! e^{\frac{t}{n+1}} \cosh^{n-1}\left(\frac{t}{n+1}\right) \sinh\left(\frac{t}{n+1}\right)\Big|_{t^k}. \qquad (2.47)
\end{aligned}
$$

□

Example 2.15. Assuming $\tilde{\Xi} = -\frac{1}{3} - \frac{2}{3}\vec{x} \in \mathcal{C}\ell_{0,2}$ as indicated by Mathematica output in Example 2.13, Mathematica calculations reveal the following values of σ_N:

N= 1 $\sigma_N{}^2$= 1.33333 σ_N= 1.1547

N= 9 $\sigma_N{}^2$= 11.3264 σ_N= 3.36547

N= 17 $\sigma_N{}^2$= 21.3264 σ_N= 4.61805

N= 25 $\sigma_N{}^2$= 31.3264 σ_N= 5.597

N= 33 $\sigma_N{}^2$= 41.3264 σ_N= 6.42856

N= 41 $\sigma_N{}^2$= 51.3264 σ_N= 7.16424

N= 49 $\sigma_N{}^2$= 61.3264 σ_N= 7.83112

N= 57 $\sigma_N{}^2$= 71.3264 σ_N= 8.4455

N= 65 $\sigma_N{}^2$= 81.3264 σ_N= 9.01811

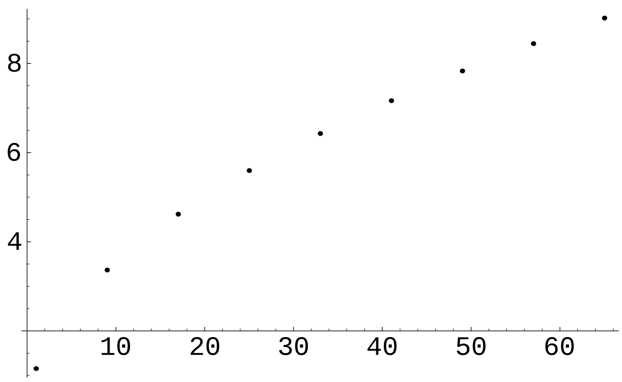

Figure 4: Standard deviation of $\Xi_N - \tilde{\tilde{\Xi}}$ in $\mathcal{C}\ell_{0,2}$ for selected values of N.

2.2 The distribution of Ψ

The distribution of the random variable Ψ is defined by

$$\mathbb{P}\left(\|\Psi\|^2 \le x\right) = \mathbb{P}\left(\sum_{\underline{i}\in 2^{[n]}} \langle \Psi, \mathbf{e}_{\underline{i}}\rangle^2 \le x\right). \tag{2.48}$$

Letting $\underline{i}$ range over the subsets of $[n]$, $\{\langle \Psi, \mathbf{e}_{\underline{i}}\rangle\}$ forms a collection of *dependent* random variables. Discussion will be restricted to the distribution of an arbitrary component $\langle \Psi, \mathbf{e}_{\underline{i}}\rangle$.

Because the limiting distribution of S_k is uniform on the positive and negative basis multivectors of $\mathcal{C}\ell_{p,q}$, it follows that for each $\underline{i} \in 2^{[n]}$, $\left(\langle S_k, \mathbf{e}_{\underline{i}}\rangle\right)_{k>0}$ is a sequence of random variables having values in $\{-1, 0, 1\}$ with limiting distribution

$$\lim_{k\to\infty} \mathbb{P}\left(\langle S_k, \mathbf{e}_{\underline{i}}\rangle = \beta\right) = \begin{cases} 2^{-(n+1)} & \text{if } \beta = 1 \\ 1 - 2^{-n} & \text{if } \beta = 0 \\ 2^{-(n+1)} & \text{if } \beta = -1. \end{cases} \tag{2.49}$$

It also follows that

$$\lim_{k\to\infty} \operatorname{var} \langle S_k, \mathbf{e}_{\underline{i}}\rangle = \frac{1}{2^n}, \tag{2.50}$$

and that

$$\lim_{k\to\infty} \langle\langle S_k, \mathbf{e}_{\underline{i}}\rangle\rangle = 0. \tag{2.51}$$

Convergence in distribution of the sequence $(\langle S_k, \mathbf{e}_{\underline{i}}\rangle)_{k>0}$ implies that for any $\varepsilon > 0$, there exists N_ε such that $N > N_\varepsilon$ implies

$$F_{N-N_\varepsilon}(x) - \varepsilon \leq \mathbb{P}\left(\sum_{k=N_\varepsilon}^{N} \langle S_k, \mathbf{e}_{\underline{i}}\rangle \leq x\right) \leq F_{N-N_\varepsilon}(x) + \varepsilon, \tag{2.52}$$

where writing $M = N - N_\varepsilon$ yields

$$F_M(x) = \sum_{\kappa=-M}^{\lfloor x \rfloor} f_M(\kappa). \tag{2.53}$$

Here $f_M(x)$ is the M^{th} mass function defined by

$$f_M(x) = \sum_{\substack{z=0 \\ M-|x|\equiv z \pmod 2}}^{M-|x|} \binom{M}{z} \binom{M-z}{\frac{M-z-|x|}{2}} \left(\frac{2^n-1}{2^n}\right)^z \left(\frac{1}{2^{n+1}}\right)^{M-z}, \tag{2.54}$$

which has support $\{-M, \ldots, M\}$ and gives the probability that the sum of M random variables taking values in $\{-1, 0, 1\}$ with respective probabilities $\{\frac{1}{2^{n+1}}, \frac{2^n-1}{2^n}, \frac{1}{2^{n+1}}\}$ is equal to x. The sum is over the mutually exclusive events that z of the M terms are equal to zero.

A more descriptive expression for the mass function is

$$f_M(x) = \left(\frac{1}{2^{n+1}}\right)^{|x|} \binom{M}{|x|} \sum_{z=0}^{\lfloor \frac{M-|x|}{2} \rfloor} \binom{M-|x|}{M-|x|-2z, z, z} \left(1 - \frac{1}{2^n}\right)^{M-|x|-2z} \left(\frac{1}{2^{n+1}}\right)^{2z}. \tag{2.55}$$

Fix $x \in \mathbb{R}$ and $N_\varepsilon > 0$. As $N \to \infty$, $|F_N(x) - F_{N-N_\varepsilon}(x)| \to 0$. Observing that $\sum_{k=1}^{N} \langle S_k, \mathbf{e}_{\underline{i}} \rangle = \langle \Xi_N, \mathbf{e}_{\underline{i}} \rangle$ and replacing Ξ_{N_ε} by $\tilde{\Xi}$, one finds

$$\left| \mathbb{P}\left(\left\langle \Xi_N - \tilde{\Xi}, \mathbf{e}_{\underline{i}} \right\rangle \leq x \right) - F_N(x) \right| \to 0 \tag{2.56}$$

as $N \to \infty$. Hence, the result of the following lemma:

Lemma 2.16. *Given arbitrary multi-index* $\underline{i} \in 2^{[n]}$*, the random variable* Ψ *of Theorem 2.14 has coordinate probability distribution function*

$$\mathbb{P}\left(\langle \Psi, \mathbf{e}_{\underline{i}} \rangle \leq x \right) = \lim_{N \to \infty} \mathbb{P}\left(\left\langle \frac{\Xi_N - \tilde{\Xi}}{\sigma_N}, \mathbf{e}_{\underline{i}} \right\rangle \leq x \right) = \lim_{N \to \infty} F_N(x\sigma_N), \tag{2.57}$$

where F_N *is the distribution function of* (2.2) *and* $\sigma_N = \sqrt{{\sigma_N}^2}$ *as defined in* (2.14).

Example 2.17. The distribution functions $F_N(x\sigma_N)$ are computed in $\mathcal{C}\ell_{0,2}$ and plotted for $N = 4, 16, 128, 256$ in Figure 5.

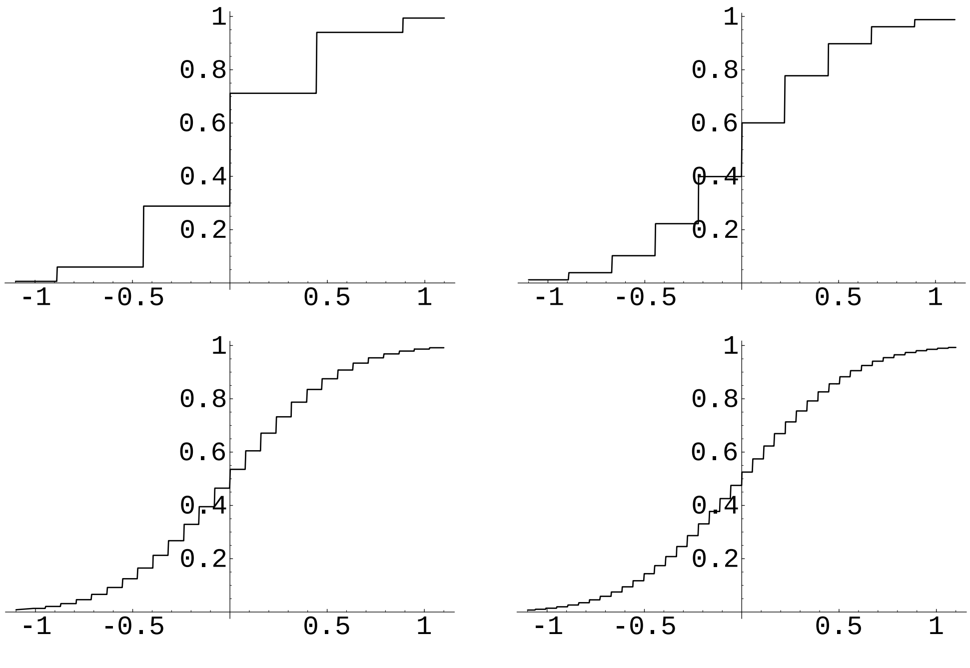

Figure 5: Distribution functions $F_4(x\sigma_4)$, $F_{16}(x\sigma_{16})$, $F_{128}(x\sigma_{128})$, and $F_{256}(x\sigma_{256})$ computed in $\mathcal{C}\ell_{0,2}$.

Example 2.18. The density function of the normal distribution $\mathcal{N}\left(0, \frac{1}{\sqrt{2\pi}\sigma_N f_N(0)}\right)$ is $\sigma_N f_N(0)e^{-\pi f_N(0)^2 x^2}$. In Figure 6, the discrete points of $\sigma_{256} f_{256}(x)$ computed in $\mathcal{C}\ell_{0,2}$ are displayed in black. The smooth curve $y = f_{256}(0)\sigma_{256}e^{-\pi f_{256}(0)^2 \sigma_{256}{}^2 x^2}$ is displayed in red.

In Figure 7, the distribution functions for 32 and 64 steps of the random walk on $\mathcal{C}\ell_{0,2}$ are overlaid with the normal distributions $\mathcal{N}\left(0, \frac{1}{\sqrt{2\pi}\sigma_{32} f_{32}(0)}\right)$ and $\mathcal{N}\left(0, \frac{1}{\sqrt{2\pi}\sigma_{64} f_{64}(0)}\right)$, respectively.

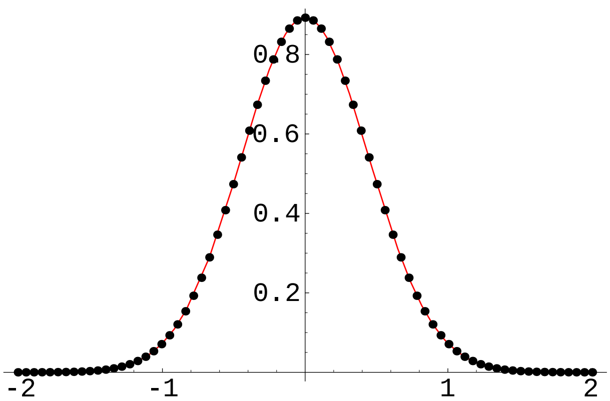

Figure 6: Discrete probability mass function $\sigma_{256} f_{256}(x\sigma_{256})$ (discrete points) and $y = \sigma_{256} f_{256}(0)e^{-\pi f_{32}(0)^2 \sigma_{256}{}^2 x^2}$ (smooth curve) computed in $\mathcal{C}\ell_{0,2}$.

2.3 Concluding remarks

This work represents one step toward a comprehensive theory of stochastic processes on Clifford algebras of arbitrary signature.

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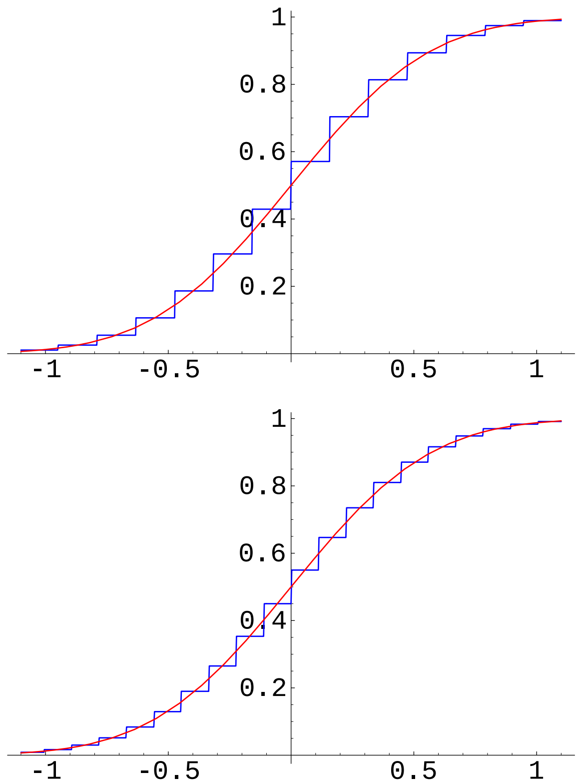

Figure 7: Distribution functions $F_{32}(\sigma_{32}x)$ and $F_{64}(\sigma_{64}x)$ (step functions) and the curves $y = \sigma_{32}f_{32}(0)\int_{-32}^{x} e^{-\pi f_{32}(0)^2\sigma_{32}{}^2t^2}\,dt$ and $y = \sigma_{64}f_{64}(0)\int_{-64}^{x} e^{-\pi f_{64}(0)^2\sigma_{64}{}^2t^2}\,dt$ computed in $\mathcal{C}\ell_{0,2}$.

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